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A NUMERICAL METHOD FOR PLASTIC LIMIT ANALYSIS OF 3-D STRUCTURES

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Abstract—Based on the upper bound theorem of plasticity, the 3-D limit analysis of rigid–perfectly plastic structures is formulated as a discrete nonlinear mathematical programming problem with only equality constraints by means of the finite element technique. The penalty function method is used to deal with the plastic incompressibility condition. A direct iterative algorithm is employed in solving this formulation. At each step of the iteration, the rigid and plastic zones are continually distinguished, the respective constraint conditions are imposed on them, and the goal function is modified appropriately. The numerical difficulties caused by the nonlinearity and nonsmoothness of the goal function and the incompressibility of plastic deformation are overcome. The limit load multiplier and the associated velocity field computed by the iteration procedure converge monotonically to the upper bounds of real solutions. The numerical procedure has been used to carry out the limit analysis for cylindrical shells with part-through slot-type defects under internal pressure. Numerical examples are given to demonstrate the applicability of the procedure.

1. INTRODUCTION

The limit analysis of structures is a very useful subdiscipline of plasticity, which can determine the load-carrying capacity of structures and provide the theoretical foundation necessary for engineering design. When a structure reaches the limit state, purely plastic strain rates will take place under constant stress distribution and the structure will be turned into a collapse mechanism. Although the classical upper and lower bound theorems of plastic limit analysis were established in the 1950s, complete analytical solutions for the limit loads are difficult to obtain for individual problems with complicated geometric forms and loading conditions. A computational approach to the limit solution is considered as the most challenging area. Rapid progress in the finite element technique and mathematical programming in the last 20 years makes it possible to develop numerical methods for the limit analysis of structures.

Two numerical approaches have been adopted to deal with the limit analysis problem. The first is to compute the limit loads by performing a series of suitable incremental elastic–plastic analyses, as has been done by Argyris (1965), Marcal and King (1967), etc. However, the step-by-step process may involve excessive computation if only the collapse load and its corresponding collapse mechanism are required for design purposes. The second approach rests on the fundamental limit theorems of plasticity and aims at computing directly the limit load multiplier by combining finite element analysis with the mathematical programming technique. It has become a main method for solving the problem of limit analysis of complicated structures. A comprehensive survey of its application to engineering plastic analysis was given by Maier and Munro (1982). Discretizing the velocity field by the finite element technique, Hayes and Marcal (1967) proposed a mathematical programming formulation for the upper bound analysis of plane stress structures. Because of the nonlinearity and nonsmoothness of the goal function, they could only solve it by the primary coordinate cycle method. Belytschko and Hodge (1970) established a nonlinear mathematical programming formulation with the von Mises yield condition as the constraint for plane stress problems by means of the equilibrium finite element, and solved it by the SUMT method. Hutala (1976) presented a unified finite element procedure for calculating upper and lower bounds to the limit load for 2-D plane structures. However, that approach introduced some stringent hypotheses and the modified exponential penalty function method used could not assure that the finite element solution converged to the real limit load. Anderheggen and Knopfel (1972), Faccioli and Vitiello (1973) and Laudiero (1972) performed the lower bound analysis as a linear programming problem concerning piecewise linearized yield surfaces. Casciaro and Cascini (1982) constructed a sequential unconstrained programming algorithm using the exponential penalty function method to remove the plastic yield condition. The computed limit load was neither the real upper bound nor the lower bound. Christiansen (1981) suggested a mixed finite element approximation to the infinite dimensional mathematical programming problem of limit analysis. Zhang *et al.* (1991) established a finite element mathematical programming formulation for the upper bound analysis and proposed an efficient algorithm for solving it. However, some difficulties existed in the application of this approach. Huh and Yang (1991) derived the dual formulation of the lower bound analysis and solved plane stress limit problems using a combined smoothing and successive approximation algorithm.

Summarizing all the works stated above, we find that some problems still exist in many algorithms available for limit analysis, such as the complexity of computational formulation, low efficiency for problem solving and the limited scope of applications, etc. Numerical limit analysis for 3-D structures is rarely performed in the present literature. Limit analysis applications are mainly concerned with plane stress–strain and axisymmetric plate/shell problems. Many numerical methods for the limit solutions are not generally appropriate for performing 3-D problems. Special considerations are required for the 3-D limit analysis problem, such as how to establish a concise computational formulation, how to present an efficient algorithm for solving it and how to deal with the numerical problems caused by the plastic incompressibility, etc. These problems have not been solved so far. Therefore, developing efficient, reliable and feasible numerical methods and their corresponding software for complicated limit analysis of practical engineering structures has been a main subject in this research field.

This paper develops a general numerical method for 3-D limit analysis problems. A finite element mathematical programming formulation for the upper bound analysis of 3-D structures is established. The incompressibility condition is dealt with successfully by the penalty function method. A direct iterative algorithm is used to solve the formulation. The monotonic convergence of the iterative process is proved.

2. UPPER BOUND FINITE ELEMENT PROCEDURE

2.1. *Finite element formulation*

Consider a 3-D, rigid–perfectly plastic body V with the boundary surface S as shown in Fig. 1. The reference surface tractions $\bar{t}_i$ $(i = 1, 2, 3)$ are prescribed on a part of the surface, S_σ, and the remaining part S_u is held in a fixed position. The body reaches the limit state

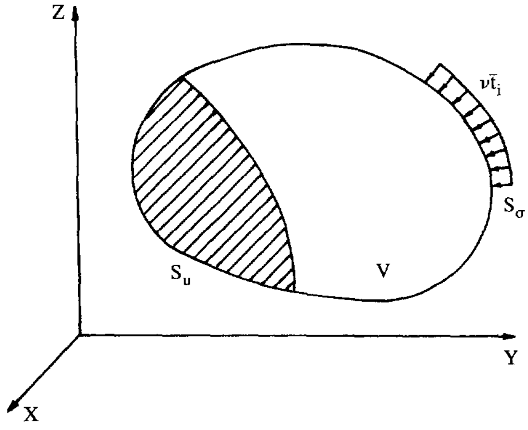


Fig. 1. 3-D rigid–perfectly plastic body loaded by surface tractions.

under the action of the proportional surface tractions $v\bar{t}_i$ (body forces are ignored), where v is a monotonically increasing load multiplier.

The 3-D body is previously assumed in the fully plastic state. By adopting the von Mises yield criterion and the upper bound theorem of plasticity, the mathematical programming formulation can be constructed as follows

$$v = \min: \sqrt{\frac{2}{3}}\sigma_s \int_V \sqrt{\varepsilon_{ij}\varepsilon_{ij}}\,\mathrm{d}V \tag{1}$$

$$s.t. \quad \int_S u_i \bar{t}_i \,\mathrm{d}S = 1\,, \quad \text{on} \quad S_\sigma \tag{2}$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})\,, \quad \text{in} \quad V \tag{3}$$

$$u_{i,i} = 0\,, \quad \text{in} \quad V \tag{4}$$

$$u_i = 0\,, \quad \text{on} \quad S_u. \tag{5}$$

In the following discussion, for convenience, the factor $\sqrt{\frac{2}{3}}\sigma_s$ of the goal function (1) is omitted temporarily.

Discretizing eqns (1)–(5) by displacement isoparametric finite elements, the strain rate vector ε_e for each element can be expressed in terms of the nodal velocity vector δ_e for each element as

$$\varepsilon_e = \mathbf{B}_e \delta_e \quad \varepsilon_e = [\varepsilon_x\, \varepsilon_y\, \varepsilon_z\, \sqrt{2}\,\varepsilon_{xy}\, \sqrt{2}\,\varepsilon_{yz}\, \sqrt{2}\,\varepsilon_{xz}]^{\mathrm{T}}, \tag{6}$$

where $\mathbf{B}_e$ is the strain matrix.

Let $\mathbf{G}_e = \mathbf{B}_e^{\mathrm{T}} \cdot \mathbf{B}_e$; the expression $\varepsilon_{ij}\varepsilon_{ij}$ may be written in matrix form as:

$$\varepsilon_{ij}\varepsilon_{ij} = \varepsilon_e^{\mathrm{T}} \cdot \varepsilon_e = \delta_e^{\mathrm{T}} \mathbf{G}_e \delta_e. \tag{7}$$

Introducing the velocity boundary condition, assembling the nodal velocity vector δ_e for each element into the global nodal velocity vector δ, extending $\mathbf{G}_e$ for each element into the global matrix $\mathbf{G}$ with the same dimension as δ and using a Gaussian integration technique, one can write the goal function (1) (i.e. the plastic dissipation work) as

$$\int_V \sqrt{\varepsilon_{ij}\varepsilon_{ij}}\,\mathrm{d}V = \sum_e^N \int_{V_e} \sqrt{\delta_e^{\mathrm{T}} \mathbf{G}_e \delta_e}\,\mathrm{d}V = \sum_{i \in I} \rho_i |J|_i \sqrt{\delta^{\mathrm{T}} \mathbf{G}_i \delta}\,, \tag{8}$$

where I is the set of all Gaussian integration points and ρ_i, $\mathbf{G}_i$ and $|J|_i$ are the integration weight, the values of the matrix $\mathbf{G}$ and the Jacobian determinant $|J|$ at the Gaussian point i, respectively.

Substituting the interpolation form of velocity field $\mathbf{u}_e = \mathbf{N}_e \delta_e$ into eqn (2), we get

$$\mathbf{F}^{\mathrm{T}} \delta = 1 \quad \mathbf{F} = \int_{S_e} \mathbf{N}^{\mathrm{T}} \bar{\mathbf{t}}_i \,\mathrm{d}S, \tag{9}$$

where $\mathbf{F}$ is the equivalent nodal load vector corresponding to the prescribed tractions and $\mathbf{N}$ is the matrix of shape functions.

Based on eqns (8) and (9), we obtain a discrete programming formulation for the upper bound analysis as follows:

$$\min : \sum_{i\in I} \rho_i |J|_i \sqrt{\delta^T \mathbf{G}_i \delta} \tag{10}$$

$$s.t. \quad \mathbf{F}^T \delta = 1. \tag{11}$$

The incompressibility condition (4) can be satisfied naturally by modifying the matrix $\mathbf{G}_i$ suitably in cases such as plane stresses and plate/shell problems. However, it requires special treatment for the upper bound analysis of 3-D problems, as discussed in the next section.

2.2. *Numerical treatment of the incompressibility condition*

The important effect of the plastic flow incompressibility on the overall solution procedure was first recognized in the classical paper of Nagtegaal *et al.* (1974). The imposition of incompressibility constraint in finite element applications is traditionally troublesome. How to deal with the incompressibility condition is one of the difficulties in the numerical upper bound analysis of 3-D structures. Two methods can be used to treat the incompressibility constraint in the plastic limit range, including the mixed finite element method and the penalty function method. Of these alternatives the penalty function method seems particularly attractive in that it has the advantage of being a single-field formulation, permitting the use of a displacement finite element code with only slight modification, and hence can be employed easily in the upper bound analysis of 3-D problems. Considering the characteristics of the computational formulation (10) and (11), we use the penalty function method to deal with the incompressibility condition.

Imposing the incompressibility constraint $\varepsilon_v = u_{i,i} = 0$ on the plastic zones by means of the penalty function method, the penalty function term is

$$\tfrac{1}{2}\alpha \int_v \varepsilon_v^2 \, \mathrm{d}V = \tfrac{1}{2}\alpha \int_v \varepsilon^T \mathbf{C} \varepsilon \, \mathrm{d}V. \tag{12}$$

Discretizing eqn (12) by the finite element technique, and substituting eqn (6) into eqn (12), we have

$$\tfrac{1}{2}\alpha \int_v \varepsilon_v^2 \, \mathrm{d}V = \tfrac{1}{2}\alpha \sum_e^N \int_{v_e} \varepsilon_e^T \mathbf{C} \varepsilon_e \, \mathrm{d}V = \tfrac{1}{2}\alpha \sum_{i\in I} \rho_i |J|_i \delta^T (\mathbf{G}_v)_i \delta, \tag{13}$$

where $\mathbf{G}_v = \mathbf{B}_e^T \mathbf{C} \mathbf{B}_e$, $\mathbf{C}$, $\mathbf{C}$ is a constant matrix and α is a sequential penalty factor which is adjusted in a suitable manner during the iterative process so as to obtain the best numerical accuracy and stability; I is the set of all plastic integration points.

Generally speaking, volumetric locking often occurs due to over-restrictions on the constraint condition when the penalty function method is used to deal with the incompressibility constraint. Locking can be avoided by employing selective-reduced integration to make the assembled volumetric stiffness matrix singular. However, our trial tests with reduced integration of the penalty function term (13) showed that fictitious values of the limit loads can be obtained. The calculated limit loads may be even smaller than the real limit loads, even though the computational formulation is based on the upper bound theorem of plasticity. Moreover, the convergence of reduced integration is much worse than that of full integration. The reason may be that when reduced integration is used, the incompressibility constraint cannot hold accurately, and the loaded body is flexibly modeled owing to the violation of the incompressiblity constraint. Since the present algorithm is aimed to obtain the upper bounds to the real limit loads, the incompressibility condition must be satisfied everywhere. Hence, it is necessary to use the same integration points for both the goal function term (8) corresponding to the plastic dissipation work and the penalty function term (13) induced by the incompressibility condition. The numerical tests by the authors showed that the proposed algorithm for the limit analysis of 3-D structures is not sensitive to the volumetric locking effect. The so-called volumetric locking does not appear when full integration is adopted.

From eqns (8), (9) and (13), we can obtain the discrete version of the 3-D upper bound analysis formulation:

$$v = \min : \sum_{i \in I} \rho_i |J|_i \sqrt{\delta^T \mathbf{G}_i \delta} \tag{14}$$

$$s.t. \quad \mathbf{F}^T \delta = 1 \tag{15}$$

$$\delta^T (\mathbf{G}_v)_i \delta = 0, \quad i \in I. \tag{16}$$

This is a nonlinear mathematical programming problem with only equality constraints. Its goal function is nonlinear and nonsmooth. This feature becomes the main difficulty in the upper bound analysis.

3. A DIRECT ITERATIVE ALGORITHM

A direct iterative algorithm is constructed to solve the proposed formulation (14)–(16). The basic characteristic of this algorithm is that at each iteration the rigid zones are distinguished from the plastic zones, the respective constraint conditions are imposed on them, and the goal function and constraint conditions are modified accordingly. The difficulties caused by undetermined rigid zones and undifferentiable goal function are overcome. The problem is reduced to an equivalent elastic one.

3.1. *Iterative procedure*

As for the formulation (14)–(16), all constraints are equalities, and the goal function is convex but not continuously differentiable and its minimum is not stationary. The nonsmoothness in the derivatives may cause trouble in the gradient or the minimization of the goal function.

We first assume that the strain at every Gauss point does not vanish, i.e.

$$\delta^T \mathbf{G}_i \delta \neq 0, \quad i \in I. \tag{17}$$

Introducing the Lagrange multiplier μ to remove $\mathbf{F}^T\delta = 1$, we get the minimization problem as follows:

$$\min : \sum_{i \in I} \rho_i |J|_i \sqrt{\delta^T \mathbf{G}_i \delta} - \mu(\mathbf{F}^T \delta - 1) \tag{18}$$

$$s.t. \quad \delta^T (\mathbf{G}_v)_i \delta = 0, \quad i \in I. \tag{19}$$

The constraint (19) can be removed simply by the penalty function method. According to the necessary conditions of minimization, we have:

$$\sum_{i \in I} \frac{\rho_i |J|_i \mathbf{G}_i \delta}{\sqrt{\delta^T \mathbf{G}_i \delta}} = \mu \mathbf{F} \tag{20}$$

$$\mathbf{F}^T \delta = 1 \tag{21}$$

$$\delta^T (\mathbf{G}_v)_i \delta = 0, \quad i \in I. \tag{22}$$

Point multiplying the two sides of eqn (20) by δ, we can find that μ is a discrete solution of the limit load multiplier v. Equations (20)–(22) are a set of nonlinear equations, and are not easy to solve directly. In general, the solution can only be approached iteratively and numerically. Hence, we construct an iterative scheme by linearizing the set of equations (20)–(22) as follows:

$$\sum_{i \in I} \frac{\rho_i |J|_i \mathbf{G}_i \delta_{k+1}}{\sqrt{\delta_k^{\mathrm{T}} \mathbf{G}_i \delta_k}} = \mu_{k+1} \mathbf{F} \tag{23}$$

$$\mathbf{F}^{\mathrm{T}} \delta_{k+1} = 1 \tag{24}$$

$$\delta_{k+1}^{\mathrm{T}} (\mathbf{G}_v)_i \delta_{k+1} = 0, \quad i \in I, \tag{25}$$

where δ_{k+1} and μ_{k+1} are the nodal velocity vector and the limit load multiplier at the $k+1$th iteration, respectively. The relation between them is

$$\mu_{k-1} = \sum_{i \in I} \frac{\rho_i |J|_i \delta_{k+1}^{\mathrm{T}} \mathbf{G}_i \delta_{k+1}}{\sqrt{\delta_k^{\mathrm{T}} \mathbf{G}_i \delta_k}}. \tag{26}$$

The iterative process of eqns (23)–(25) actually corresponds to solving a series of the minimization problems of quadratic functions as follows:

$$\min: \sum_{i \in I} \frac{\rho_i |J|_i \delta^{\mathrm{T}} \mathbf{G}_i \delta}{\sqrt{\delta_k^{\mathrm{T}} \mathbf{G}_i \delta_k}} \tag{27}$$

$$s.t. \quad \mathbf{F}^{\mathrm{T}} \delta = 1 \tag{28}$$

$$\delta^{\mathrm{T}} (\mathbf{G}_v)_i \delta = 0, \quad i \in I. \tag{29}$$

They are very similar to solving a series of relevant elastic problems.

However, in the general case, the matrix $\mathbf{G}_i$ is only positive semi-definite, such that the term $\delta_k^{\mathrm{T}} \mathbf{G}_i \delta_k$ may vanish for some nontrivial vectors δ_k. When $\delta_k^{\mathrm{T}} \mathbf{G}_i \delta_k$ vanishes, the above iterative process cannot go on normally. Hence, this case needs dealing with specially, so as to ensure each iterative step can proceed smoothly.

Prior to the $k+1$th iterative step, one should examine the strain value of every integration point one by one to find out if $\delta_k^{\mathrm{T}} \mathbf{G}_i \delta_k$ vanishes or not. So the set I of all integration points can be divided into the rigid zone point subset R_{k+1} and the plastic zone point subset P_{k+1}, i.e.

$$I = R_{k+1} \cup \mathrm{P}_{k+1} \tag{30}$$

$$R_{k+1} = \{ i \in I, \quad \delta_k^{\mathrm{T}} \mathbf{G}_i \delta_k = 0 \} \tag{31}$$

$$P_{k-1} = \{ i \in I, \quad \delta_k^{\mathrm{T}} \mathbf{G}_i \delta_k \neq 0 \}. \tag{32}$$

The determinations of set R_{k+1} and P_{k-1} are essential for removing those at the rigid state from the sum of the integration points of the goal function so as to ensure that the next iteration can proceed smoothly. In the meantime, physically, we can also learn the distribution of the plastic and rigid zones during the iterative process by eqns (30)–(32).

For this reason, the constraint condition is imposed on the rigid zone points as follows:

$$\delta^{\mathrm{T}} \mathbf{G}_i \delta = 0, \quad i \in R_{k+1}. \tag{33}$$

For the points in R_{k+1}, the constraint condition (33) can guarantee the satisfaction of the incompressibility condition. For the points in P_{k+1}, the following constraint condition is imposed so as to satisfy the incompressibility constraint:

$$\delta^{\mathrm{T}}(\mathbf{G}_v)_i\delta = 0\,, \quad i \in P_{k+1}. \tag{34}$$

Considering eqns (33) and (34), we get the modified iterative form

$$\sum_{i \in P_{k+1}} \frac{\rho_i |J|_i \mathbf{G}_i \delta_{k+1}}{\sqrt{\delta_k^{\mathrm{T}} \mathbf{G}_i \delta_k}} = \mu_{k+1}\mathbf{F} \tag{35}$$

$$\mathbf{F}^{\mathrm{T}}\delta_{k+1} = 1 \tag{36}$$

$$\delta_{k+1}^{\mathrm{T}}(\mathbf{G}_v)_i\delta_{k+1} = 0\,, \quad i \in P_{k+1} \tag{37}$$

$$\delta_{k+1}^{\mathrm{T}}\mathbf{G}_i\delta_{k+1} = 0\,, \quad i \in R_{k+1}. \tag{38}$$

Then we obtain a series of minimization formulations of quadratic functions corresponding to eqns (35)–(38) as follows:

$$\min: \sum_{i \in P_{k+1}} \frac{\rho_i |J|_i \delta^{\mathrm{T}} \mathbf{G}_i \delta}{\sqrt{\delta_k^{\mathrm{T}} \mathbf{G}_i \delta_k}} \tag{39}$$

$$s.t. \quad \mathbf{F}^{\mathrm{T}}\delta = 1 \tag{40}$$

$$\delta^{\mathrm{T}}(\mathbf{G}_v)_i\delta = 0\,, \quad i \in P_{k+1} \tag{41}$$

$$\delta^{\mathrm{T}}\mathbf{G}_i\delta = 0\,, \quad i \in R_{k+1}. \tag{42}$$

This modified iteration formulation is always valid. The rigid zones are recognized from the plastic zones, and consequently the goal function and the constraint conditions are modified. The difficulties caused by the nonsmooth goal function and the incompressibility condition are overcome.

3.2. *The iterative process*

The algorithm proceeds as follows:

Step 0 (*choosing an initial estimate*). The selection of the initial velocity field hardly matters at all to the convergence of iteration. It can be proved that from any initial trial solution, the subsequent iterations are locked in a certain convex hull that contains the exact solution of the problem. For convenience, here we solve the initial value as a minimization problem as follows:

$$\min: \sum_{i \in I} \rho_i |J|_i \delta^{\mathrm{T}} \mathbf{G}_i \delta \tag{43}$$

$$s.t. \quad \mathbf{F}^{\mathrm{T}}\delta = 1 \tag{44}$$

$$\delta^{\mathrm{T}}(\mathbf{G}_v)_i\delta = 0\,, \quad i \in I. \tag{45}$$

Introducing the constraint conditions as eqns (44) and (45) to the goal function by the Lagrange multiplier technique and penalty function method, respectively. Based on the minimization conditions, the above problem is equivalent to solving a set of linear algebraic equations:

$$\sum_{i\in I}\rho_i|J|_i\mathbf{G}_i\delta_0+\sum_{i\in I}\alpha_0^i(\mathbf{G}_v)_i\delta_0=\lambda_0\mathbf{F} \tag{46}$$

$$\mathbf{F}^{\mathrm{T}}\delta_0=1, \tag{47}$$

where λ_0 is the Lagrange multiplier and α_0^i is the initial penalty factor.

From eqns (46) and (47), we can solve the initial velocity field δ_0. Substituting δ_0 back into eqn (14), we obtain the initial estimate of the limit load multiplier:

$$v_0=\sum_{i\in I}\rho_i|J|_i\sqrt{\delta_0^{\mathrm{T}}\mathbf{G}_i\delta_0}\,. \tag{48}$$

The computation shows this procedure can produce close estimates of the actual solution.

Step k + 1. (1) As stated in Section 3.1, we examine the strain value of every integration point to find out if it vanishes or not, and determine the rigid zone point set R_{k+1} and plastic zone point set P_{k+1}.

(2) Solution of the minimization problem:

$$\min:\sum_{i\in P_{k+1}}\frac{\rho_i|J|_i\delta^{\mathrm{T}}\mathbf{G}_i\delta}{\sqrt{\delta_k^{\mathrm{T}}\mathbf{G}_i\delta_k}} \tag{49}$$

$$s.t.\quad \mathbf{F}^{\mathrm{T}}\delta=1 \tag{50}$$

$$\delta^{\mathrm{T}}(\mathbf{G}_v)_i\delta=0\,,\quad i\in P_{k+1} \tag{51}$$

$$\delta^{\mathrm{T}}\mathbf{G}_i\delta=0\,,\quad i\in R_{k+1}. \tag{52}$$

The constraint conditions in eqns (50), (51) and (52) can be removed by the Lagrange multiplier technique and the penalty function method, respectively. According to the minimization conditions, the above problem is equivalent to solving the set of linear algebraic equations as follows:

$$\sum_{i\in P_{k+1}}\frac{\rho_i|J|_i\mathbf{G}_i\delta_{k+1}}{\sqrt{\delta_k^{\mathrm{T}}\mathbf{G}_i\delta_k}}+\sum_{i\in R_{k+1}}\beta_{k+1}^i\,\mathbf{G}_i\delta_{k+1}+\sum_{i\in P_{k+1}}\alpha_{k+1}^i(\mathbf{G}_v)_i\delta_{k+1}=\lambda_{k+1}\mathbf{F} \tag{53}$$

$$\mathbf{F}^{\mathrm{T}}\delta_{k+1}=1, \tag{54}$$

where α_{k+1}^i and β_{k+1}^i are the penalty factors at the $k+1$th iteration, and λ_{k+1} is the Lagrange multiplier at the $k+1$th iteration.

From eqns (53) and (54), we get the iterative velocity value δ_{k+1}. Substituting δ_{k+1} back into eqns (26) and (14), we have

$$\mu_{k+1}=\sum_{i\in P_{k+1}}\frac{\rho_i|J|_i\delta_{k+1}^{\mathrm{T}}\mathbf{G}_i\delta_{k+1}}{\sqrt{\delta_k^{\mathrm{T}}\mathbf{G}_i\delta_k}} \tag{55}$$

$$v_{k+1}=\sum_{i\in I}\rho_i|J|_i\sqrt{\delta_{k+1}^{\mathrm{T}}\mathbf{G}_i\delta_{k+1}}\,. \tag{56}$$

(3) Solution of the sets of linear equations. During the process of each iteration, the equivalent programming formulation of eqns (43)–(45) or (49)–(52) has the general form

$$\min : \delta_{k+1}^{\mathrm{T}} \mathbf{A}(\delta_k) \delta_{k+1} \tag{57}$$

$$s.t. \quad \mathbf{F}^{\mathrm{T}} \delta_{k+1} = 1, \quad k = 0, 1, 2, \ldots, \tag{58}$$

where $\mathbf{A}$ is a symmetric positive definite matrix whose elements are functions of δ_k, and assembled by the different linear combinations of $\mathbf{G}_i$ and $(\mathbf{G}_v)_i$ corresponding to every integration point.

At each iterative step $k+1$, we treat $\mathbf{A}$ as a constant matrix and solve the unconstrained quadratic programming problem, i.e.

$$\min : \delta^{\mathrm{T}} \mathbf{A} \delta - 2\lambda(\mathbf{F}^{\mathrm{T}} \delta - 1), \tag{59}$$

where λ is the Lagrangian multiplier. The solution of eqn (59) is

$$\lambda = \frac{1}{\mathbf{F}^{\mathrm{T}} \mathbf{v}}, \quad \delta = \lambda \mathbf{v}, \quad \delta^{\mathrm{T}} \mathbf{A} \delta = \lambda, \tag{60}$$

where $\mathbf{v}$ is the solution of the sparse linear system $\mathbf{A}\mathbf{v} = \mathbf{F}$.

(4) Convergence criteria. The iterative process is terminated as soon as

$$\frac{|\nu_{k+1} - \nu_k|}{\nu_k} \leqslant \mathrm{VOL1} \tag{61}$$

$$\frac{\|\delta_{k+1} - \delta_k\|}{\|\delta_k\|} \leqslant \mathrm{VOL2}, \tag{62}$$

where VOL1 and VOL2 are the desired accuracies of the caculation.

If inequalities (61) and (62) cannot be satisfied, the procedure will modify the penalty factors and then return to proceed with the next step of the iteration.

(5) Choosing the penalty factor. The penalty factor sequence $\{\alpha_k\}$ is properly modified by the convergence of constraint conditions during the iterative process, i.e.

$$\alpha_{k+1}^i = \begin{cases} \gamma \alpha_k^i & \text{when } |(\varepsilon_v^i)_k| \geqslant \theta\, |(\varepsilon_v^i)_{k-1}| \text{ and } \|(\varepsilon_{mn}^i)_k\|^2 \geqslant \theta\, \|(\varepsilon_{mn}^i)_{k-1}\|^2 \\ \alpha_k^i & \text{otherwise} \end{cases}, \tag{63}$$

where k is the iterative step, i is the integration point, and $0 \leqslant \theta \leqslant 1$, $\gamma \geqslant 1$ are constant parameters. The step length of the sequence $\{\alpha_k\}$ is controlled by modifying γ and θ so that we can obtain the desired solutions satisfying the accuracy by adjusting α_k at every iterative step. Adjusting the penalty factors during the iterative process can assure the numerical stability of the iteration algorithm, accelerate the convergence rate of the iterative process and make the velocity field δ_k satisfy the constraint conditions well. According to our numerical tests, the selections of $\theta \in [0.1, 0.4]$ and $\gamma \in [6, 10]$ can obtain the best effects.

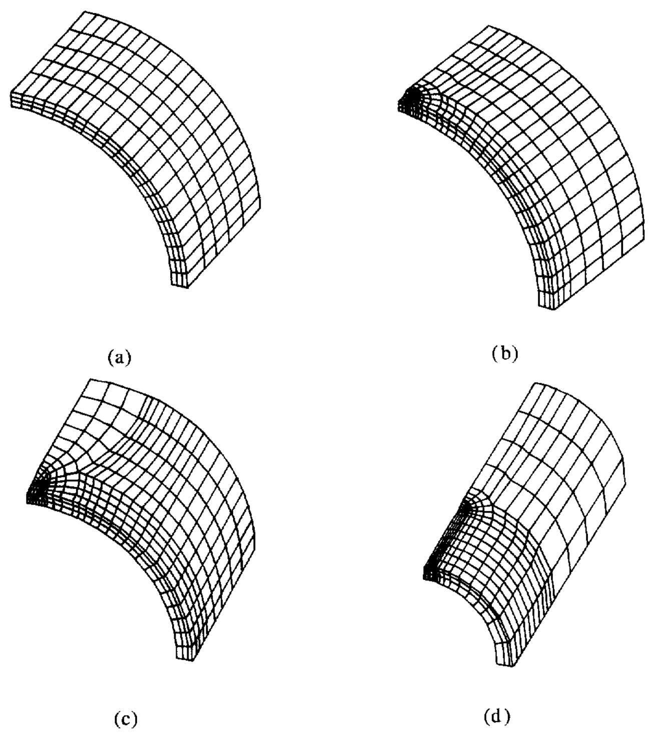
Through the above iteration process, a limit load multiplier sequence $\{\nu_k\}$ is obtained. If the constraints (51) and (52) hold exactly, it can be proved that $\{\nu_k\}$ is a monotonically decreasing sequence, and the limit load multiplier and the associated velocity field produced by the iteration process converge to the upper bounds of real solutions.

4. APPLICATIONS

The proposed numerical method has been coded in a computer program operating on a 486 personal computer. Numerical experiments were performed in order to optimize the numerical efficiency and accuracy by taking into account different operational parameters such as mesh density, the rigid zones check criterion, the convergence criterion, the penalty factor, etc. The said algorithm has been applied to the limit analysis for cylindrical shells with part-through slots in order to prove its effectiveness. The numerical results are compared with existing solutions and the results of detailed 3-D elastic–plastic finite element analysis.

A part-through slot is a common defect configuration on the surface of pressure vessels. It can be formed by corrosion or produced by abrading surface cracks. Part-through slots may have greater or lesser effects on the load-carrying capacity of pressure vessels. The evaluation of these effects caused by part-through slots presents a problem to the engineers. In principle, this problem belongs to the regime of plastic limit analysis of 3-D structures. Using the presented algorithm, the numerical limit pressures have been computed for cylindrical shells with slot-type defects on the surface. These defects include part-through spherical, ellipsoidal and rectangular slots. The failure modes are also studied.

First of all, the limit pressures are computed by the proposed algorithm for cylindrical shells without slot defects and with various dimensions and shapes of part-through slots. We also employ the ADINA program to compute the same problems by 3-D incremental elastic–plastic analysis. The radius ratio *kk* (i.e. the external to internal radius) is 1.20. The cylinder thickness is 20 mm. The yield stress σ_s is 200 MPa. Considering the symmetry, we discretize a quadrant of a cylindrical shell by 3-D eight-node isoparametric finite elements. The corresponding displacement constraints are imposed on the symmetric boundary. For the cylindrical shell with slot defects, owing to different shapes and sizes of slots, the finite element mesh should be chosen appropriately so that the distribution of the elements is uniform and focused around slots. In terms of different slot parameters, the total number of elements employed ranges from 400 to 1000, and the total number of nodes adopted ranges from 700 to 1200. The finite element mesh adopted is shown in Fig. 2. The above calculated results are compared with the analytical solutions in Table 1. According to Table 1, the results obtained by this paper agree well with those of the ADINA program and analytical solutions, while the proposed algorithm costs much less computer time than the incremental method. The calculated values for the limit pressures are the upper bounds very close to the real limit loads. Table 2 represents the convergence of the values of the



(a) (b) (c) (d)

Fig. 2. The finite element mesh: (a) cylinder without defect; (b) cylinder with a part-through spherical slot; (c) cylinder with a part-through ellipsoidal slot; (d) cylinder with a part-through longitudinal slot.

Table 1. Comparison of the limit pressures (MPa) obtained by different methods

Structure	This paper	ADINA	Analytical solution
Fig. 2(a)	42.463	41.925	42.107
Fig. 2(b)	39.824	39.202	
Fig. 2(c)	35.716	35.075	
Fig. 2(d)	28.594	28.001	

Table 2. The iterative values of limit pressures (MPa)

Iterative step	Fig. 2(a)	Fig. 2(b)
0	53.842	52.614
1	49.935	49.822
2	46.712	47.354
3	44.636	45.427
4	43.182	43.964
5	42.629	42.803
6	42.520	41.745
7	42.463	40.986
8		40.623
9		40.325
10		40.116
11		39.957
12		39.865
13		39.824

limit pressures for the cylinders as shown in Fig. 2(a) and (b). The iterative values of limit pressures decrease progressively as the iterative step increases, and finally converge to the real limit loads. The numerical limit load may generally be found in a reasonable number of iterative steps. The iterative process shows stable convergence and computational efficiency. This algorithm is particularly suitable for computing the limit loads of 3-D complicated structures.

Kitching and Zarrabi (1981, 1982) considered part-through rectangular slots theoretically and experimentally. They computed a lower bound to the limit pressure of a cylindrical shell with a part-through rectangular slot using a linear optimization technique and did experimental investigations to estimate plastic limit pressures for comparison with the calculated results. However, because they introduced some assumptions to simplify the limit analysis, the calculated limit pressures are considerably below real limit pressures. This paper has computed an upper bound to the limit pressure for a cylindrical shell with part-through rectangular slots for the geometric parameters corresponding to some theoretical investigations by Kitching and Zarrabi (1981). The comparison of the numerical results with the theoretical lower bound from Kitching and Zarrabi (1981) and that due to approximate analysis of Ruiz (1978) is shown in Fig. 3. In Fig. 3, Ω denotes the ratio of

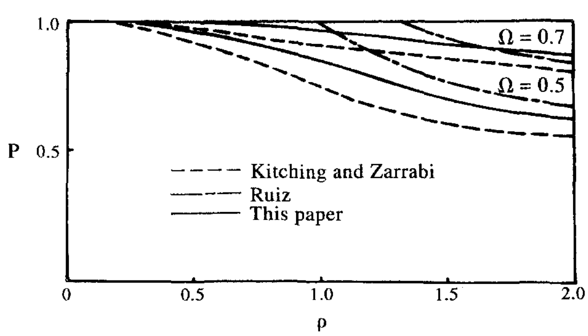
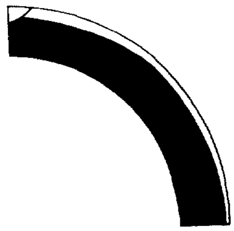
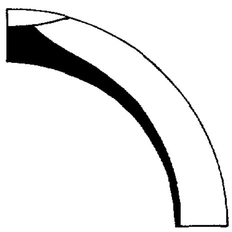


Fig. 3. Comparison of the results from this paper with those from Kitching and Zarrabi (1981) and Ruiz (1978).



(a)



(b)

Fig. 4. The failure modes at the limit state: (a) global collapse; (b) local collapse.

the ligament thickness to the cylinder thickness, ρ represents a geometric parameter which combines the slot geometry and the cylinder dimensions, and P is the dimensionless limit pressure defined as the ratio of the limit pressure for a cylinder with a slot to that for a cylinder of the same dimensions without a slot. Figure 3 shows the results obtained in this paper (upper bound) are greater, and sometimes reasonably greater than those of Kitching and Zarrabi (1981) (lower bound) and are lower than those from Ruiz (1978); the results from Kitching and Zarrabi (1981) may be too conservative. The combination of the results from this paper and Kitching and Zarrabi seems to imply a safe and nonconservative design method.

The cylindrical shell with small slot dimensions and a large ligament is almost in a full yielding state when the internal pressure reaches the limit load. The local plastic hinge will not generally be produced around the slot. In this case, the effect of a slot on the limit carrying capacity of cylinder is insignificant, which is mainly the weakening of the net section. The global collapse of the cylinder will take place at the limit state. The plastic region at the limit state is shown in Fig. 4(a).

For the cylindrical shell with large slot dimensions and a small ligament, plastic yielding first occurs at the bottom of the slot, and expands rapidly along the longitudinal direction on the surface of the slot where the stress level is relatively high, while the plastic region extends slowly along the circumferential and radial direction of the cylindrical shell. As the internal pressure increases, new plastic regions are formed on the inside surface around the slot. With the plastic regions extending continuously, two rigid regions are formed on the inside surface around the slot and the outside surface far from the slot, respectively. When the rigid regions on the inside surface around the slot yield, a plastic hinge is formed around the slot. The cylinder reaches the limit state and becomes a collapse mechanism. The ligament of the slot bulges outward. In this case, local leakage will occur within the slot for the cylindrical shell. Figure 4(b) represents the collapse mechanism corresponding to the local failure mode.

5. CONCLUSIONS

In the present paper, we have proposed a numerical method for the limit analysis of 3-D structures. The method is built on sound physical, mathematical and computational foundations. An upper bound to the limit load can be obtained by a direct iteration algorithm. By distinguishing the rigid zones from plastic zones gradually, imposing the respective constraints on them, and modifying the goal function and the constraint conditions accordingly, the numerical difficulties caused by the nonlinearity and nonsmoothness of the goal function and the incompressibility of plastic deformation are overcome. Every step of the iterative process is equivalent to solving a relevant elastic problem. It has been proved by the authors that the iterative process is convergent monotonically even without a careful and elaborate initial value. Unlike the incremental method, the error of the iterative solution is not accumulative.

This numerical method can be implemented in the upper bound analysis of complicated 3-D structures. It was successfully applied to compute the limit pressures and investigate the failure modes for cylindrical shells with various part-through slots under the action of internal pressures. The numerical applications confirm the validity and usefulness of the present method. It can be also generalized for 3-D structures subjected to combined constant loads and proportional loads. In this case, a set of surface loads acting on a region S of the boundary consists of "live" loads $\mathbf{F}^{\mathrm{l}}$ which are described by a given distribution but a common unknown load multiplier to be minimized and "dead" loads $\mathbf{F}^{\mathrm{d}}$ whose value and distribution is assigned. The programming formulation can be written as:

$$v = \min : \sum_{i \in I} \rho_i |J|_i \sqrt{\delta^{\mathrm{T}} \mathbf{G}_i \delta} - (\mathbf{F}^{\mathrm{d}})^{\mathrm{T}} \delta \tag{64}$$

$$s.t. \quad (\mathbf{F}^{\mathrm{l}})^{\mathrm{T}} \delta = 1 \tag{65}$$

$$\delta^{\mathrm{T}} (\mathbf{G}_v)_i \delta = 0, \quad i \in I. \tag{66}$$

The iterative process is similar to that presented in the preceding Section 3.2.

The proposed numerical method possesses high efficiency, little computation and good numerical stability. The numerical examples show the computational results are satisfactory. It can be easily implemented by means of current existing displacement finite element codes for elastic analysis. Because of the use of the displacement finite element technique, this method is applicable to a wide variety of complicated 3-D structures and loadings.

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